

THE NUMBER OF SIMPLE MODULES FOR THE HECKE ALGEBRAS OF TYPE $G(r, p, n)$

(with an appendix by Xiaoyi Cui)

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Dedicated to Professor Gus Lehrer on the occasion of his sixtieth birthday

ABSTRACT. We derive a parameterization of simple modules for the cyclotomic Hecke algebras of type $G(r, p, n)$ over field of any characteristic coprime to p . We give explicit formulas for the number of simple modules over these cyclotomic Hecke algebras.

1. INTRODUCTION

Let r, p, d and n be positive integers such that $pd = r$. Let K be a field such that K contains a primitive p -th root of unity ε . Let x_1, \dots, x_d be invertible elements in K . Let $q \neq 1$ be an invertible element in K . Let $\mathcal{H}_K(r, n)$ be the unital K -algebra with generators T_0, T_1, \dots, T_{n-1} and relations

$$\begin{aligned} (T_0^p - x_1^p)(T_0^p - x_2^p) \cdots (T_0^p - x_d^p) &= 0, \\ T_0 T_1 T_0 T_1 &= T_1 T_0 T_1 T_0, \\ (T_i + 1)(T_i - q) &= 0, \quad \text{for } 1 \leq i \leq n-1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad \text{for } 1 \leq i \leq n-2, \\ T_i T_j &= T_j T_i, \quad \text{for } 0 \leq i < j-1 \leq n-2. \end{aligned}$$

Let $\mathcal{H}_K(r, p, n)$ be the subalgebra of $\mathcal{H}_K(r, n)$ generated by the elements $T_0^p, T_u := T_0^{-1} T_1 T_0, T_1, T_2, \dots, T_{n-1}$. This algebra is called the cyclotomic Hecke algebra of type $G(r, p, n)$, which was introduced in [3], [6] and [8]. It includes Hecke algebras of type A , type B and type D as special cases. The algebra $\mathcal{H}_K(r, 1, n)$ is called the Ariki-Koike algebra. These algebras are conjecturally related to Lusztig's induced characters in the modular representation of finite reductive groups over field of non-defining characteristic (see [8]).

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The representation of Ariki–Koike algebra (e.g., $\mathcal{H}_K(r, n)$) is well understood by the work of [1], [2], [10] and [11]. Let \mathcal{P}_n be the set of r -multipartitions of n . Let $\vec{Q} := (Q_1, \dots, Q_r)$ be a fixed arbitrary permutation of

$$\left(\underbrace{x_1, x_1\varepsilon, \dots, x_1\varepsilon^{p-1}}_{p \text{ terms}}, \dots, \dots, \underbrace{x_d, x_d\varepsilon, \dots, x_d\varepsilon^{p-1}}_{p \text{ terms}} \right).$$

We use Q to denote the underlying unordered multiset (allowing repetitions) of \vec{Q} . For any $\lambda \in \mathcal{P}_n$, let \tilde{S}_Q^λ be the Specht module defined in [10]. There is a naturally defined bilinear form \langle, \rangle on \tilde{S}_Q^λ . Let $\tilde{D}_Q^\lambda = \tilde{S}_Q^\lambda / \text{rad}\langle, \rangle$. By [10], the set $\{\tilde{D}_Q^\lambda \mid \lambda \in \mathcal{P}_n, \tilde{D}_Q^\lambda \neq 0\}$ forms a complete set of pairwise non-isomorphic simple $\mathcal{H}_K(r, n)$ -modules. By [2] and [11], $\tilde{D}_Q^\lambda \neq 0$ if and only if λ is a Kleshchev r -multipartition of n with respect to (q, \vec{Q}) .

When $q \neq 1$ is a root of unity, Jacon gives in [25] another parameterization of simple $\mathcal{H}_K(r, n)$ -modules via FLOTW r -partitions. As an application, a parameterization of simple $\mathcal{H}_K(r, p, n)$ -modules is obtained in [16]. The parameterization results in both [16] and [25] are valid only when $K = \mathbb{C}$ (the complex number field). In [20] and [22], using a different approach, we obtain a parameterization of simple $\mathcal{H}_K(p, p, n)$ -modules which is valid over field of any characteristic coprime to p , and we give explicit formula for the number of simple modules of $\mathcal{H}_K(p, p, n)$. In this paper, combining the results in [16] with the results and ideas in [22], we derive a parameterization as well as explicit formula for the number of simple $\mathcal{H}_K(r, p, n)$ -modules which is valid over field of any characteristic coprime to p . These results generalize the earlier results in [14], [16], [18], [19], [20], [21], [22] and [31], and was already announced in [23]. At the end of this paper there is an appendix given by Xiaoyi Cui who fixes a gap in the proof of [15, (2.2)]. We remark that the latter result is crucial to both the present paper and the paper [16].

Throughout this paper, $q \neq 1$ is an invertible element in K . Let e be the smallest positive integer such that $1 + q + q^2 + \dots + q^{e-1} = 0$ in K ; or ∞ if no such positive integer exists. We fix elements $z_1, \dots, z_s \in K^\times$, such that $z_i z_j^{-1} \notin q^{\mathbb{Z}}$, $\forall i \neq j$, and for each $1 \leq i \leq r$, $Q_i \in z_j q^{\mathbb{Z}}$ for some $1 \leq j \leq s$.

2. KLESHCHEV r -MULTIPARTITIONS AND KLESHCHEV'S GOOD LATTICE

Let λ be an r -multipartition. The diagram of λ is the set

$$[\lambda] = \{(i, j, s) \mid 1 \leq j \leq \lambda_i^{(s)} \text{ for } 1 \leq s \leq r\}.$$

The elements of $[\lambda]$ are called the nodes of λ . Given any two nodes $\gamma = (a, b, c), \gamma' = (a', b', c')$ of λ , say that γ is *below* γ' , or γ' is *above* γ with respect to the *Kleshchev order*, if either $c > c'$ or $c = c'$ and $a > a'$. With

respect to the $(r+1)$ -tuple (q, Q_1, \dots, Q_r) , the residue of a node $\gamma = (a, b, c)$ is defined to be $\text{res}(\gamma) := Q_c q^{b-a} \in K$. We call γ a $\text{res}(\gamma)$ -node. The node $\gamma = (a, \lambda_a^{(c)}, c)$ is called a removable node of λ if $\lambda_a^{(c)} > \lambda_{a+1}^{(c)}$. In that case, $\lambda \setminus \{\gamma\}$ is again an r -multipartition, and we call γ an addable node of $\lambda \setminus \{\gamma\}$. For a fixed residue $x \in K$, say that a removable x -node γ of λ is a normal x -node, if whenever η is an addable x -node of λ which is below γ , there are more removable x -node of λ between η and γ than there are addable x -nodes. If γ is the highest normal x -node of λ , we say that γ is a good x -node. If λ is obtained from μ by removing a good x -node of μ , we write that $\lambda \xrightarrow{x} \mu$.

Definition 2.1. ([7], [28]) Suppose $n \geq 0$. The set \mathcal{K}_n of Kleshchev r -multipartitions of n with respect to (q, Q_1, \dots, Q_r) is defined inductively as follows:

- (1) $\mathcal{K}_0 := \left\{ \underline{\emptyset} := (\underbrace{\emptyset, \dots, \emptyset}_{r \text{ copies}}) \right\};$
- (2) $\mathcal{K}_{n+1} := \left\{ \mu \in \mathcal{P}_{n+1} \mid \lambda \xrightarrow{x} \mu \text{ for some } \lambda \in \mathcal{K}_n \text{ and some } x \in K \right\}.$

Let $\mathcal{K} := \sqcup_{n \geq 0} \mathcal{K}_n$. The Kleshchev's good lattice with respect to (q, Q_1, \dots, Q_r) is, by definition, the infinite graph whose vertices are the Kleshchev r -multipartitions with respect to (q, Q_1, \dots, Q_r) and whose arrows are given by $\lambda \xrightarrow{x} \mu$. For any $\lambda \in \mathcal{K}$ and any $1 \leq i \leq s, 0 \leq j \leq e-1$, we define

$$\begin{aligned} \tilde{f}_{i,j} \lambda &:= \begin{cases} \lambda \cup \{\gamma\}, & \text{if } \gamma \text{ is a good } (z_i q^j)\text{-node of } \lambda \cup \{\gamma\}; \\ 0, & \text{otherwise,} \end{cases} \\ \tilde{e}_{i,j} \lambda &:= \begin{cases} \lambda \setminus \{\gamma\}, & \text{if } \gamma \text{ is a good } (z_i q^j)\text{-node of } \lambda; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

By [2] and [11], for any $\lambda \in \mathcal{P}_n$, $\tilde{D}_{\vec{Q}}^\lambda \neq 0$ if and only if $\lambda \in \mathcal{K}_n$.

Definition 2.2. Let τ be the K -algebra automorphism of $\mathcal{H}_K(r, n)$ which is defined on generators by $\tau(T_1) = T_0^{-1} T_1 T_0, \tau(T_i) = T_i$, for any $i \neq 1$. Let σ be the nontrivial K -algebra automorphism of $\mathcal{H}_K(r, n)$ which is defined on generators by $\sigma(T_0) = \varepsilon T_0, \sigma(T_i) = T_i$, for any $1 \leq i \leq n-1$.

Note that if M is a simple $\mathcal{H}_K(r, n)$ -module, then M^σ is again a simple $\mathcal{H}_K(r, n)$ -module.

Definition 2.3. Let h be the automorphism of \mathcal{K}_n which is defined by $(\tilde{D}_{\vec{Q}}^\lambda)^\sigma \cong \tilde{D}_{\vec{Q}}^{h(\lambda)}$.

Clearly, $h^p = \text{id}$. In particular, we get an action of the cyclic group C_p on \mathcal{K}_n given as follows:

$$\tilde{D}_{\vec{Q}}^{\sigma^k \cdot \lambda} \cong (\tilde{D}_{\vec{Q}}^\lambda)^{\sigma^k}, \quad \forall k \in \mathbb{Z}.$$

Let \sim_σ be the corresponding equivalence relation on \mathcal{K}_n . That is, $\lambda \sim_\sigma \mu$ if and only if $\lambda = g \cdot \mu$ for some $g \in C_p$. For each $\lambda \in \mathcal{K}_n / \sim_\sigma$, let C_λ be the stabilizer of λ in C_p . The following results are basically followed from [15, (2.2)] and [16, Lemma 2.2] (see [20, (5.4), (5.5), (5.6)] for an independent proof in the case where $r = p$). Unfortunately, the proof of [15, (2.2)] given there contains a gap (as noted in [26]). That is, in the 10th line of Page 527, Genet's claim about the determinant of the representing matrix is generally false. Since the result [15, (2.2)] is crucial for both the present paper and the paper [16], we include at the end of this paper an appendix given by Xiaoyi Cui who fixes the gap.

Lemma 2.4. *Suppose that $\mathcal{H}_K(r, p, n)$ is split over K .*

1) *Let $\tilde{D}_{\mathbb{Q}}^\lambda$ be any given irreducible $\mathcal{H}_K(r, n)$ -module and D be an irreducible $\mathcal{H}_K(r, p, n)$ -submodule of $\tilde{D}_{\mathbb{Q}}^\lambda$. Let d_0 be the smallest positive integer such that $D \cong (D)^{\tau^{d_0}}$. Then $1 \leq d_0 \leq p$, and $k := p/d_0$ is the smallest positive integer such that $\tilde{D}_{\mathbb{Q}}^\lambda \cong (\tilde{D}_{\mathbb{Q}}^\lambda)^{\sigma^k}$, and*

$$\tilde{D}_{\mathbb{Q}}^\lambda \downarrow_{\mathcal{H}_K(r, p, n)} \cong D \oplus D^\tau \oplus \cdots \oplus (D)^{\tau^{d_0-1}}.$$

2) *The set $\left\{ D^{\lambda, 0}, D^{\lambda, 1}, \dots, D^{\lambda, |C_\lambda|-1} \mid \lambda \in \mathcal{K}_n / \sim_\sigma \right\}$ forms a complete set of pairwise non-isomorphic simple $\mathcal{H}_K(r, p, n)$ -modules, where for each $\lambda \in \mathcal{K}_n / \sim_\sigma$, $D^{\lambda, 0}$ is an irreducible $\mathcal{H}_K(r, p, n)$ submodule of $\tilde{D}_{\mathbb{Q}}^\lambda$, and $D^{\lambda, i} = (D^{\lambda, 0})^{\tau^i}$ for $i = 0, 1, \dots, |C_\lambda| - 1$.*

Proof. By [15] and [16], the results in this lemma hold if K is the complex number field. Furthermore, it is easy to see that all the arguments in [15, (2.2)] and [16, Lemma 2.2] are actually valid for any algebraically closed field K of characteristic coprime to p . As a direct consequence of [16, Lemma 2.2] and Frobenius reciprocity, the statements in this lemma are valid whenever K is an algebraically closed field of characteristic coprime to p . Now using the fact (see [17]) that every simple module for the algebra $\mathcal{H}_K(r, n)$ is always absolutely simple, it follows that these statements remain valid whenever $\mathcal{H}_K(r, p, n)$ is split over K . \square

Therefore, the problem on classifying simple $\mathcal{H}_K(r, p, n)$ -modules reduces to the problem of determining the automorphism h .

3. FLOTW r -PARTITIONS AND FLOTW'S GOOD LATTICE

Throughout this section, we assume that $e < \infty$.

For each integer c with $1 \leq c \leq r$, we fix an integer $0 \leq v_c \leq e - 1$ such that $Q_c = z_i q^{v_c}$ for some integers i with $1 \leq i \leq s$. Throughout this section,

we make the following assumption:

the order on \vec{Q} are chosen such that, whenever $Q_{i_1}, Q_{i_2}, \dots, Q_{i_s}$ is a q -orbit in Q , where $i_1 < i_2 < \dots < i_s$, we have $0 \leq v_{i_1} \leq v_{i_2} \leq \dots \leq v_{i_s} < e$.

Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ be an r -multipartition, or equivalently (see [25]), an r -partition. Given any two nodes $\gamma = (a, b, c), \gamma' = (a', b', c')$ of λ , say that γ is *below* γ' , or γ' is *above* γ with respect to the *FLOTW order*, if whenever $Q_c, Q_{c'}$ are in a single q -orbit, then either $b - a + v_c > b' - a' + v_{c'}$ or $b - a + v_c = b' - a' + v_{c'}$ and $c < c'$. Note that the FLOTW order does depend on the choice of the elements $\{z_1, \dots, z_s\}$ (which we have fixed at the end of Section 1). In a similar way as before (see Section 2), we have the notions of normal x -nodes and good x -nodes with respect to the FLOTW order. If λ is obtained from μ by removing a good x -node of μ , we write that $\lambda \xrightarrow{x} \mu$. If $Q_{i_1}, Q_{i_2}, \dots, Q_{i_s}$ form a q -orbit in \vec{Q} , then we call the multipartition $(\lambda^{(i_1)}, \lambda^{(i_2)}, \dots, \lambda^{(i_s)})$ the restriction of λ to that q -orbit.

Definition 3.1. ([12]) *Suppose $n \geq 0$. Let $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)}) \in \mathcal{P}_n$. If Q is a single q -orbit, then λ is a FLOTW r -partition of n with respect to (q, \vec{Q}) if and only if:*

(1) *for all $1 \leq j \leq r-1$ and $i = 1, 2, \dots$, we have*

$$\lambda_i^{(j)} \geq \lambda_{i+v_{j+1}-v_j}^{(j+1)}, \quad \lambda_i^{(r)} \geq \lambda_{i+e+v_1-v_r}^{(1)};$$

(2) *for any $k \geq 0$, among the residues appearing at the right ends of the length k rows of λ , at least one element of $\{z_1, z_1q, \dots, z_1q^{e-1}\}$ does not occur.*

In general, if \vec{Q} is a disjoint union of several q -orbits, then λ is a FLOTW r -partition of n with respect to (q, \vec{Q}) if and only if with respect to each q -orbit of \vec{Q} , the restriction of λ to that q -orbit satisfies the above two conditions.

By [12], one can also give a recursive definition (like (2.1)) of FLOTW r -partition by using the procedure of adding good nodes. Note that, at the moment, we do not have a non-recursive definition for Kleshchev r -multipartition except for $r \leq 2$ (see [5], [9]).

Let \mathcal{F}_n be the set of all the FLOTW r -partitions of n with respect to (q, Q_1, \dots, Q_r) . Let $\mathcal{F} := \sqcup_{n \geq 0} \mathcal{F}_n$. The *FLOTW's good lattice* (w.r.t. (q, Q_1, \dots, Q_r)) is, by definition, the infinite graph whose vertices are the FLOTW r -partitions with respect to (q, Q_1, \dots, Q_r) and whose arrows are given by $\lambda \xrightarrow{x} \mu$. For any $\lambda \in \mathcal{F}$ and any $1 \leq i \leq s, 0 \leq j \leq e-1$, we define

$$\begin{aligned} \tilde{f}_{i,j} \circ \lambda &:= \begin{cases} \lambda \cup \{\gamma\}, & \text{if } \gamma \text{ is a good } (z_i q^j)\text{-node of } \lambda \cup \{\gamma\}; \\ 0, & \text{otherwise,} \end{cases} \\ \tilde{e}_{i,j} \circ \lambda &:= \begin{cases} \lambda \setminus \{\gamma\}, & \text{if } \gamma \text{ is a good } (z_i q^j)\text{-node of } \lambda; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

When $s = 1$, i.e., the parameters $\{Q_1, \dots, Q_r\}$ are in a single q -orbit, both the Kleshchev's good lattice and FLOTW's good lattice provide realizations of the crystal graph of certain irreducible integrable highest weight module over the quantum affine algebra $U'_v(\widehat{\mathfrak{sl}}_e)$.

Lemma 3.2. ([25, (4.1)]) *There is a unique bijection $\kappa : \mathcal{K} \sqcup \{0\} \rightarrow \mathcal{F} \sqcup \{0\}$, such that, $\kappa(0) = 0$, $\kappa(\underline{\emptyset}) = \underline{\emptyset}$ and for any $\lambda \in \mathcal{K}$, and any $1 \leq i \leq s, 0 \leq j \leq e - 1$,*

$$\kappa(\tilde{f}_{i,j}\lambda) = \tilde{f}_{i,j} \circ \kappa(\lambda), \quad \kappa(\tilde{e}_{i,j}\lambda) = \tilde{e}_{i,j} \circ \kappa(\lambda),$$

and for any $n \geq 0$ and any $\lambda \in \mathcal{K}_n$, the following identity holds in the Grothendieck group of finite dimensional $\mathcal{H}_K(r, n)$ -modules,

$$\left[\tilde{S}_{\underline{Q}}^{\kappa(\lambda)} \right] = \left[\tilde{D}_{\underline{Q}}^{\lambda} \right] + \sum_{\substack{\mu \in \mathcal{K}_n \\ a(\kappa(\mu)) < a(\kappa(\lambda))}} d_{\kappa(\lambda), \mu} \left[\tilde{D}_{\underline{Q}}^{\mu} \right],$$

where $\lambda \in \mathcal{K}_n$, $d_{\kappa(\lambda), \mu} \in \mathbb{Z}^{\geq 0}$ and $a(?)$ is the a -function defined in [16, (2.4.6)].

4. A DESCRIPTION OF h

The following are the first two main results in this paper, which yield (by Lemma 2.4) a parameterization of simple modules for the cyclotomic Hecke algebras of type $G(r, p, n)$ over field of any characteristic coprime to p .

Theorem 4.1. *The automorphism h does not depend on the choice of the base field K as long as $\mathcal{H}_K(r, p, n)$ is split over K and K contains a primitive p -th root of unity.*

Proof. This is proved by using the same argument as in the Appendix of [21]. \square

Theorem 4.2. *Let $\lambda \in \mathcal{K}_n$ be a Kleshchev r -multipartition of n with respect to (q, Q_1, \dots, Q_r) . Then, if $\underline{\emptyset} \xrightarrow{r_1} \cdot \xrightarrow{r_2} \dots \xrightarrow{r_n} \lambda$ is a path from $\underline{\emptyset}$ to λ in Kleshchev's good lattice with respect to (q, Q_1, \dots, Q_r) , then the sequence*

$$\underline{\emptyset} \xrightarrow{\varepsilon r_1} \cdot \xrightarrow{\varepsilon r_2} \dots \xrightarrow{\varepsilon r_n} h(\lambda)$$

also defines a path in Kleshchev's good lattice with respect to (q, Q_1, \dots, Q_r) , and it connects $\underline{\emptyset}$ to $h(\lambda)$.¹

Note that the definition of h depends on the chosen order on \underline{Q} . To stress this point, we had better use the notation $h_{\underline{Q}}$ instead of h . Before giving the proof of Theorem 4.2, we note the following result.

¹One can compare this theorem with [19, (1.5)], [22, (3.11)] and [27, Theorem 7.1].

Lemma 4.3. *Let $\vec{Q}'' = (Q_1'', \dots, Q_r'')$ be an arbitrary permutation of \vec{Q} . Then Theorem 4.2 is valid for $h_{\vec{Q}}$ if and only if it is valid for $h_{\vec{Q}''}$.*

Proof. Let $\mathcal{K}_{\vec{Q}}$ (resp., $\mathcal{K}_{\vec{Q}''}$) be the set of Kleshchev r -multipartitions with respect to (q, \vec{Q}) (resp., with respect to (q, \vec{Q}'')). There is a bijection θ from $\mathcal{K}_{\vec{Q}}$ onto $\mathcal{K}_{\vec{Q}''}$, such that $\tilde{D}_{\vec{Q}}^\lambda \cong \tilde{D}_{\vec{Q}''}^{\theta(\lambda)}$, for any $n \geq 0$ and any $\lambda \in \mathcal{K}_{\vec{Q}} \cap \mathcal{P}_n$. Note that

$$\tilde{D}_{\vec{Q}}^{h_{\vec{Q}}(\lambda)} = \left(\tilde{D}_{\vec{Q}}^\lambda \right)^\sigma \cong \left(\tilde{D}_{\vec{Q}''}^{\theta(\lambda)} \right)^\sigma = \tilde{D}_{\vec{Q}''}^{h_{\vec{Q}''}(\theta(\lambda))}.$$

It follows that $\theta(h_{\vec{Q}}(\lambda)) = h_{\vec{Q}''}(\theta(\lambda))$.

Since the operator $\tilde{f}_{i,j}$ can also be defined as taking socle of the j -restriction of module (see [4]), it follows that the bijection θ satisfies

$$\theta(\tilde{f}_{i,j}\lambda) = \tilde{f}_{i,j}\theta(\lambda),$$

for any $\lambda \in \mathcal{K}_{\vec{Q}}$ and any $1 \leq i \leq s, 0 \leq j \leq e-1$, from which the lemma follows at once. \square

The above lemma allows us to feel free to choose appropriate order on \vec{Q} . This remaining part of this section is devoted to the proof of Theorem 4.2.

First, we need to classify the q -orbits in \vec{Q} (cf. [16, Section 2.4.2]). Two elements Q_i, Q_j are said to be in the same q -orbit if $Q_i \in Q_j q^{\mathbb{Z}}$; while two elements Q_i, Q_j are said to be in the same (ε, q) -orbit if $Q_i \in Q_j \varepsilon^{\mathbb{Z}} q^{\mathbb{Z}}$. Note that for any $1 \leq i, j \leq d$, $x_i \in x_j \varepsilon^{\mathbb{Z}} q^{\mathbb{Z}}$ if and only if $x_i \varepsilon^a \in (x_j \varepsilon^b) \varepsilon^{\mathbb{Z}} q^{\mathbb{Z}}$ for some (and hence any) $0 \leq a, b \leq p-1$. Therefore, we can split \vec{Q} into a disjoint union of d' subsets (for some integer $1 \leq d' \leq d$):

$$\vec{Q} = \vec{Q}^{[1]} \sqcup \vec{Q}^{[2]} \sqcup \dots \sqcup \vec{Q}^{[d']},$$

such that two elements Q_i, Q_j are in the same (ε, q) -orbit if and only if they belong to the same subset $\vec{Q}^{[i]}$ for some $1 \leq i \leq d'$. Then for each $1 \leq i \leq d'$, $|\vec{Q}^{[i]}| = pd_i$ for some integer d_i . Without loss of generality, we can assume that

$$\begin{aligned} \vec{Q}^{[1]} &= \left\{ x_j \varepsilon^b \mid 1 \leq j \leq d_1, 0 \leq b \leq p-1 \right\}, \\ \vec{Q}^{[2]} &= \left\{ x_j \varepsilon^b \mid d_1 + 1 \leq j \leq d_1 + d_2, 0 \leq b \leq p-1 \right\}, \\ &\vdots \\ \vec{Q}^{[d']} &= \left\{ x_j \varepsilon^b \mid d - d_{d'} + 1 \leq j \leq d, 0 \leq b \leq p-1 \right\}. \end{aligned}$$

Note that for each $\vec{Q}^{[i]}$ and each positive integer n_i , one can naturally associate with a cyclotomic Hecke algebra of type $G(pd_i, p, n_i)$ with parameters $(q, \vec{Q}^{[i]})$. Let $\vec{Q}^{[i]}$ be an ordered pd_i -tuple obtained by fixing an order on $\vec{Q}^{[i]}$, then we have an automorphism $h_{\vec{Q}^{[i]}}$ on the set of Kleshchev pd_i -multipartitions of n_i with respect to $(q, \vec{Q}^{[i]})$ which is defined in the same

way as in the case of $G(r, p, n)$. By some abuse of notation, we use \vec{Q} to denote $(\vec{Q}^{[1]}, \dots, \vec{Q}^{[d']})$, the concatenation of ordered tuples. Let $\lambda \in \mathcal{K}_n$ be a Kleshchev r -multipartition of n with respect to (q, \vec{Q}) . For each $1 \leq i \leq d'$, let

$$\lambda^{[i]} := \left(\lambda^{(pd_1 + \dots + pd_{i-1} + 1)}, \dots, \lambda^{(pd_1 + \dots + pd_i)} \right),$$

$$n_i := |\lambda^{[i]}|.$$

Lemma 4.4. *With the notations as above, we have that*

$$h_{\vec{Q}}(\lambda) = \left(h_{\vec{Q}^{[1]}}(\lambda^{[1]}), \dots, h_{\vec{Q}^{[d']}}(\lambda^{[d']}) \right).$$

In particular, the Kleshchev r -multipartition (with respect to (q, \vec{Q})) $h_{\vec{Q}}(\lambda)$ is as described in Theorem 4.2 if and only if for each $1 \leq i \leq d'$, the Kleshchev pd_i -multipartition (with respect to $(q, \vec{Q}^{[i]})$) $h_{\vec{Q}^{[i]}}(\lambda^{[i]})$ is as described in Theorem 4.2.

Proof. For each integer $n \geq 0$, let $\mathcal{H}_n^{\text{aff}}$ be the affine Hecke algebra of size n as defined in [22, Definition 5.1]. By [22, Corollary 5.6], we know that

$$D_{\vec{Q}}^{\lambda} \cong \text{Ind}_{\mathcal{H}_{n_1}^{\text{aff}} \otimes \dots \otimes \mathcal{H}_{n_{d'}}^{\text{aff}}}^{\mathcal{H}_n^{\text{aff}}} \left(D_{\vec{Q}^{[1]}}^{\lambda^{[1]}} \otimes \dots \otimes D_{\vec{Q}^{[d']}}^{\lambda^{[d']}} \right).$$

By [22, Lemma 2.4], we know that

$$\left(\tilde{D}_{\vec{Q}}^{\lambda} \right)^{\sigma} \cong \tilde{D}_{(\varepsilon \vec{Q}^{[1]}, \varepsilon \vec{Q}^{[2]}, \dots, \varepsilon \vec{Q}^{[d']})}^{\lambda}.$$

Therefore,

$$\begin{aligned} \left(\tilde{D}_{\vec{Q}}^{\lambda} \right)^{\sigma} &\cong \tilde{D}_{(\varepsilon \vec{Q}^{[1]}, \varepsilon \vec{Q}^{[2]}, \dots, \varepsilon \vec{Q}^{[d']})}^{\lambda} \\ &\cong \text{Ind}_{\mathcal{H}_{n_1}^{\text{aff}} \otimes \mathcal{H}_{n_2}^{\text{aff}} \otimes \dots \otimes \mathcal{H}_{n_{d'}}^{\text{aff}}}^{\mathcal{H}_n^{\text{aff}}} \left(D_{\varepsilon \vec{Q}^{[1]}}^{\lambda^{[1]}} \otimes D_{\varepsilon \vec{Q}^{[2]}}^{\lambda^{[2]}} \otimes \dots \otimes D_{\varepsilon \vec{Q}^{[d']}}^{\lambda^{[d']}} \right) \\ &\cong \text{Ind}_{\mathcal{H}_{n_1}^{\text{aff}} \otimes \mathcal{H}_{n_2}^{\text{aff}} \otimes \dots \otimes \mathcal{H}_{n_{d'}}^{\text{aff}}}^{\mathcal{H}_n^{\text{aff}}} \left(D_{\vec{Q}^{[1]}}^{h_{\vec{Q}^{[1]}}(\lambda^{[1]})} \otimes D_{\vec{Q}^{[2]}}^{h_{\vec{Q}^{[2]}}(\lambda^{[2]})} \otimes \dots \otimes D_{\vec{Q}^{[d']}}^{h_{\vec{Q}^{[d']}}(\lambda^{[d']})} \right) \\ &\cong \tilde{D}_{\vec{Q}}^{(h_{\vec{Q}^{[1]}}(\lambda^{[1]}), h_{\vec{Q}^{[2]}}(\lambda^{[2]}), \dots, h_{\vec{Q}^{[d']}}(\lambda^{[d']})}), \end{aligned}$$

from which the lemma follows at once. \square

The above lemma allows us to assume without loss of generality that all the elements Q_1, \dots, Q_r are in a single (ε, q) -orbit. *Henceforth, we assume that all the parameters in \vec{Q} are in a single (ε, q) -orbit.* Recall that we have assumed that $q \neq 1$ from the very beginning. The following two lemmas are useful in our discussion.

Lemma 4.5. *Let $0 \neq a \in K$. Let $aQ = \{aQ_1, \dots, aQ_r\}$. Let σ_a be the isomorphism from $\mathcal{H}_{r,n}^K(q, aQ)$ onto $\mathcal{H}_{r,n}^K(q, Q)$ which is defined on generators by $\sigma_a(T_0) = aT_0$ and $\sigma_a(T_i) = T_i$ for any $1 \leq i \leq n-1$. Let \vec{Q} be an ordered*

r -tuple which is obtained by fixing an order on \mathbf{Q} . Then for each $\lambda \in \mathcal{P}_n$, there are $\mathcal{H}_{r,n}^K(q, a\mathbf{Q})$ -module isomorphisms

$$\left(\tilde{S}_{\mathbf{Q}}^\lambda\right)^{\sigma_a} \cong \tilde{S}_{a\mathbf{Q}}^\lambda, \quad \left(\tilde{D}_{\mathbf{Q}}^\lambda\right)^{\sigma_a} \cong \tilde{D}_{a\mathbf{Q}}^\lambda,$$

where $\overrightarrow{a\mathbf{Q}}$ denotes the ordered r -tuple which is obtained from $\overrightarrow{\mathbf{Q}}$ by multiplying a on each component. In particular, $\tilde{D}_{\mathbf{Q}}^\lambda \neq 0$ if and only if $\tilde{D}_{a\mathbf{Q}}^\lambda \neq 0$. Moreover, for each $\lambda \in \mathcal{K}_n$, we have $\mathbf{h}_{\overrightarrow{\mathbf{Q}}}(\lambda) = \mathbf{h}_{\overrightarrow{a\mathbf{Q}}}(\lambda)$.

Proof. This follows directly from the definition of $\tilde{S}_{\mathbf{Q}}^\lambda$ and $\tilde{D}_{\mathbf{Q}}^\lambda$. \square

Lemma 4.6. ([22, Lemma 3.5]) *Let K be a field which contains a primitive p -th root of unity ε . Suppose $p = dk$, where $p, d, k \in \mathbb{N}$, $\xi \in K$ is a primitive d -th root of unity. Then there exists a primitive p -th root of unity $\zeta \in K$ such that $\zeta^k = \xi$.*

Using the above two lemmas, we can divide the proof of Theorem 4.2 into the following two cases:

Case 1. $q^\mathbb{Z} \cap \varepsilon^\mathbb{Z} = \{1\}$, $\overrightarrow{\mathbf{Q}} = (\overrightarrow{\mathbf{Q}}^{[1]}, \dots, \overrightarrow{\mathbf{Q}}^{[p]})$, where for each $1 \leq j \leq p$,

$$\overrightarrow{\mathbf{Q}}^{[j]} = (\varepsilon^{j-1}q^{v_1}, \dots, \varepsilon^{j-1}q^{v_d}),$$

for some integers $0 \leq v_1, \dots, v_d \leq e-1$.

In this case, for each $1 \leq i \leq p$, let

$$\lambda^{[i]} := \left(\lambda^{(d(i-1)+1)}, \dots, \lambda^{(di)}\right), \quad n_i := |\lambda^{[i]}|.$$

Applying, in turn, [22, Lemma 2.4, Corollary 5.6] and [32, (5.12)], we have that

$$\begin{aligned} \left(\tilde{D}_{\mathbf{Q}}^\lambda\right)^\sigma &\cong \tilde{D}_{(\varepsilon\overrightarrow{\mathbf{Q}}^{[1]}, \varepsilon\overrightarrow{\mathbf{Q}}^{[2]}, \dots, \varepsilon\overrightarrow{\mathbf{Q}}^{[p]})}^\lambda \cong \tilde{D}_{(\overrightarrow{\mathbf{Q}}^{[2]}, \dots, \overrightarrow{\mathbf{Q}}^{[p-1]}, \overrightarrow{\mathbf{Q}}^{[1]})}^\lambda \\ &\cong \text{Ind}_{\mathcal{H}_{n_1}^{\text{aff}} \otimes \dots \otimes \mathcal{H}_{n_{p-1}}^{\text{aff}} \otimes \mathcal{H}_{n_p}^{\text{aff}}}^{\mathcal{H}_n^{\text{aff}}} \left(D_{\overrightarrow{\mathbf{Q}}^{[2]}}^{\lambda^{[1]}} \otimes \dots \otimes D_{\overrightarrow{\mathbf{Q}}^{[p]}}^{\lambda^{[p-1]}} \otimes D_{\overrightarrow{\mathbf{Q}}^{[1]}}^{\lambda^{[p]}} \right) \\ &\cong \text{Ind}_{\mathcal{H}_{n_p}^{\text{aff}} \otimes \mathcal{H}_{n_1}^{\text{aff}} \otimes \dots \otimes \mathcal{H}_{n_{p-1}}^{\text{aff}}}^{\mathcal{H}_n^{\text{aff}}} \left(D_{\overrightarrow{\mathbf{Q}}^{[1]}}^{\lambda^{[p]}} \otimes D_{\overrightarrow{\mathbf{Q}}^{[2]}}^{\lambda^{[1]}} \otimes \dots \otimes D_{\overrightarrow{\mathbf{Q}}^{[p]}}^{\lambda^{[p-1]}} \right) \\ &\cong \tilde{D}_{\overrightarrow{\mathbf{Q}}}^{(\lambda^{[p]}, \lambda^{[1]}, \dots, \lambda^{[p-1]})}, \end{aligned}$$

It follows that

$$(4.7) \quad \mathbf{h}_{\overrightarrow{\mathbf{Q}}}(\lambda) = (\lambda^{[p]}, \lambda^{[1]}, \dots, \lambda^{[p-1]}).$$

Note that $\overrightarrow{\mathbf{Q}}^{[1]}, \dots, \overrightarrow{\mathbf{Q}}^{[p]}$ are p different q -orbits. Therefore, in this case Theorem 4.2 follows easily from (4.7).

Case 2. $\overrightarrow{\mathbf{Q}} = (\overrightarrow{\mathbf{Q}}^{[1]}, \dots, \overrightarrow{\mathbf{Q}}^{[k]})$, where $p = d_0k$, q is a primitive $d_0\ell$ -th root of unity, $q^\ell = \varepsilon^k$ is a primitive d_0 -th root of unity, and $1 \leq k < p$ is the smallest positive integer such that $\varepsilon^k \in q^\mathbb{Z}$, and for each $1 \leq j \leq k$,

$$\begin{aligned} \overrightarrow{\mathbf{Q}}^{[j]} &= (\varepsilon^{j-1}q^{v_1}, \dots, \varepsilon^{j-1}q^{v_d}, \varepsilon^{k+j-1}q^{v_1}, \dots, \varepsilon^{k+j-1}q^{v_d}, \dots, \\ &\quad \varepsilon^{(d_0-1)k+j-1}q^{v_1}, \dots, \varepsilon^{(d_0-1)k+j-1}q^{v_d}), \end{aligned}$$

where $0 \leq v_1 \leq v_2 \leq \dots \leq v_d < l$ are some integers independent of j . In particular, in each q -orbit $\overrightarrow{Q}^{[j]}$, we have (compare this with our assumption in the third paragraph in Section 3)

$$0 \leq v_1 \leq v_2 \leq \dots \leq v_d < l + v_1 \leq l + v_2 \leq \dots \leq l + v_d < \dots \\ < (d_0 - 1)l + v_1 \leq (d_0 - 1)l + v_2 \leq \dots \leq (d_0 - 1)l + v_d < e.$$

Note that, in this case by assumption $e = d_0 l < \infty$. We actually have two different approaches. The first one is based on the same arguments in [22, Section 4], we leave the details to the interested readers. In this paper, we adopt a second approach, which is based on [16, Proposition 2.10]) and the following two results. Note also that we have fixed an order of the parameters in \mathbf{Q} . This order is important in the following lemma.

Lemma 4.8. *We keep the same assumption as in Case 2, and take $z_i = \varepsilon^{i-1}$ for each integer i with $1 \leq i \leq k = s$. Then the notion of FLOTW r -partition with respect to (q, \overrightarrow{Q}) is well-defined. For any FLOTW r -partition λ with respect to (q, \overrightarrow{Q}) , and any two nodes $\gamma = (a, b, c)$, $\gamma' = (a', b', c')$ of λ with the same residue,*

$$b - a > b' - a' \iff \gamma \text{ is below } \gamma' \text{ with respect to the FLOTW order.}$$

Proof. Let $\gamma = (a, b, c), \gamma' = (a', b', c')$ be two nodes of λ with the same residue. Then $Q_c q^{b-a} = Q_{c'} q^{b'-a'}$. By our assumption on the r -tuple \overrightarrow{Q} . We have that

$$Q_c = \varepsilon^{j-1+c_1 k} q^{v_{c_2}}, \quad Q_{c'} = \varepsilon^{j-1+c'_1 k} q^{v_{c'_2}},$$

for some integers $1 \leq j \leq k$, $0 \leq c_1, c'_1 \leq d_0 - 1$, $1 \leq c_2, c'_2 \leq d$. Therefore,

$$b - a + c_1 l + v_{c_2} = b' - a' + c'_1 l + v_{c'_2} + l e$$

for some integer l .

Suppose $b - a > b' - a'$. By assumption,

$$|(c_1 l + v_{c_2}) - (c'_1 l + v_{c'_2})| < (d_0 - 1)l + l = e.$$

It follows that we must have $l \geq 0$. If $l > 0$, then

$$b - a + c_1 l + v_{c_2} > b' - a' + c'_1 l + v_{c'_2},$$

which implies that γ is below γ' with respect to the FLOTW order; while if $l = 0$, then we must have

$$0 < (b - a) - (b' - a') < e, \quad -e < (c_1 l + v_{c_2}) - (c'_1 l + v_{c'_2}) < 0.$$

Note that $|v_{c_2} - v_{c'_2}| < l$. It follows that either $c_1 < c'_1$ or $c_1 = c'_1$ and $c_2 < c'_2$. By the definition of our \overrightarrow{Q} , we deduce that $c < c'$. Hence γ is again below γ' with respect to the FLOTW order.

Conversely, if γ is below γ' with respect to the FLOTW order, it is also easy to deduce that $b - a > b' - a'$. \square

Let ω be the permutation on $\{1, 2, \dots, r\}$ which is defined by

$$\begin{aligned} xd_0d + y &\mapsto (x+1)d_0d + y, \quad \forall 0 \leq x < k-1, 1 \leq y \leq d_0d, \\ (k-1)d_0d + \hat{x} &\mapsto d + \hat{x}, \quad \forall 1 \leq \hat{x} \leq d_0d - d, \\ r - d + \hat{y} &\mapsto \hat{y}, \quad \forall 1 \leq \hat{y} \leq d. \end{aligned}$$

By the definition of our \vec{Q} (see the second paragraph above Lemma 4.8), it is easy to see that $\varepsilon\vec{Q} = (Q_{\omega(1)}, \dots, Q_{\omega(r)})$. For any $\mu = (\mu^{(1)}, \dots, \mu^{(r)}) \in \mathcal{P}_n$, we define $\omega(\mu) := (\mu^{(\omega^{-1}(1))}, \dots, \mu^{(\omega^{-1}(r))})$. Recall that (see Lemma 3.2) the map κ restricts to a bijection from the set \mathcal{K}_n of Kleshchev r -multipartitions of n with respect to (q, Q_1, \dots, Q_r) to the set \mathcal{F}_n of FLOTW r -partitions of n with respect to (q, Q_1, \dots, Q_r) .

Lemma 4.9. *Assume that $e < \infty$. Let $\lambda \in \mathcal{K}_n$ be a Kleshchev r -multipartition of n with respect to (q, Q_1, \dots, Q_r) . Then, if $\underline{\emptyset} \xrightarrow{r_1} \cdot \xrightarrow{r_2} \dots \xrightarrow{r_n} \kappa(\lambda)$ is a path from $\underline{\emptyset}$ to $\kappa(\lambda)$ in FLOTW's good lattice with respect to (q, Q_1, \dots, Q_r) , then the sequence*

$$\underline{\emptyset} \xrightarrow{\varepsilon r_1} \cdot \xrightarrow{\varepsilon r_2} \dots \xrightarrow{\varepsilon r_n} \omega(\kappa(\lambda))$$

also defines a path in FLOTW's good lattice with respect to (q, Q_1, \dots, Q_r) , and it connects $\underline{\emptyset}$ to $\omega(\kappa(\lambda))$.

Proof. By [16, Proposition 2.10], we know that for any FLOTW r -partition μ , $\omega(\mu)$ is also a FLOTW r -partition. Now let μ be a given FLOTW r -partition. Lemma 4.8 implies that $\gamma = (a, b, c)$ is a good x -node of μ if and only if $\omega(\gamma) := (a, b, \omega(c))$ is a good εx -node of $\omega(\mu)$, from which the lemma follows immediately. \square

We remark that, in the special case where $r = p$ and $\varepsilon = q^l$, Lemma 4.8 and 4.9 are proved in [24, (4.3.A)].

Proof of Theorem 4.2 in Case 2: By [16, Proposition 2.10] (see also the second line in Page 16 of [16]), we know that for any Kleshchev r -multipartition λ , $\kappa(h(\lambda)) = \omega(\kappa(\lambda))$. Now Theorem 4.2 in Case 2 follows from Lemma 3.2 and Lemma 4.9. \square

We have the following result (compare [22, Theorem 3.8]).

Proposition 4.10. *Assume that $e < \infty$, $p = d_0k$, q is a primitive d_0l -th root of unity, $q^l = \varepsilon^k$ is a primitive d_0 -th root of unity, and $1 \leq k < p$ is the smallest positive integer such that $\varepsilon^k \in q^{\mathbb{Z}}$. For each $1 \leq j \leq k$, let*

$$\begin{aligned} \vec{Q}^{[j]} &= (\varepsilon^{j-1}q^{v_1}, \dots, \varepsilon^{j-1}q^{v_d}, \varepsilon^{k+j-1}q^{v_1}, \dots, \varepsilon^{k+j-1}q^{v_d}, \dots, \dots, \\ &\quad \varepsilon^{(d_0-1)k+j-1}q^{v_1}, \dots, \varepsilon^{(d_0-1)k+j-1}q^{v_d}), \end{aligned}$$

Let $\vec{Q} := (\vec{Q}^{[1]}, \dots, \vec{Q}^{[k]})$, the concatenation of ordered tuples. Let $\lambda = (\lambda^{[1]}, \dots, \lambda^{[k]}) \in \mathcal{K}_n$, where for each $1 \leq i \leq k$,

$$\lambda^{[i]} = (\lambda^{((i-1)d_0d+1)}, \dots, \lambda^{((i-1)d_0d+d_0d)}).$$

Then

$$h(\lambda) = (h'(\lambda^{[k]}), \lambda^{[1]}, \dots, \lambda^{[k-1]}),$$

where h' is an automorphism on the set of Kleshchev $d_0 d$ -multipartition of n_k with respect to (q, \vec{Q}^\vee) , where

$$\vec{Q}^\vee := (q^{v_1}, \dots, q^{v_d}, \varepsilon^k q^{v_1}, \dots, \varepsilon^k q^{v_d}, \dots, \dots, \varepsilon^{(d_0-1)k} q^{v_1}, \dots, \varepsilon^{(d_0-1)k} q^{v_d}),$$

h' is defined (similar to h) in the context of the cyclotomic Hecke algebra $\mathcal{H}_{q, \vec{Q}^\vee}(d_0 d, d_0, n_k)$, where $n_k = |\lambda^{[k]}|$.

Proof. Note that $\vec{Q}^{[1]}, \dots, \vec{Q}^{[k]}$ are k different q -orbits. By [11, (4.11)], the assumption that $\lambda \in \mathcal{K}_n$ implies that for each $1 \leq i \leq k$, $\lambda^{[i]}$ is a Kleshchev $d_0 d$ -multipartition with respect to $(q, \vec{Q}^{[i]})$. In particular, this in turn implies that

$$(h'(\lambda^{[k]}), \lambda^{[1]}, \dots, \lambda^{[k-1]}) \in \mathcal{K}_n.$$

By the same reasoning, it is easy to see that we can find a path of the following form

$$\begin{aligned} \underline{\emptyset} = (\underbrace{\emptyset, \dots, \emptyset}_{r \text{ copies}}) &\xrightarrow{r_1} \dots \xrightarrow{r_{n_1}} (\lambda^{[1]}, \underbrace{\emptyset, \dots, \emptyset}_{r - d_0 d \text{ copies}}) \xrightarrow{r_{n_1+1}} \dots \\ &\xrightarrow{r_{n_1+n_2}} (\lambda^{[1]}, \lambda^{[2]}, \underbrace{\emptyset, \dots, \emptyset}_{r - 2d_0 d \text{ copies}}) \xrightarrow{r_{n_1+n_2+1}} \dots \xrightarrow{r_n} \lambda \end{aligned}$$

in Kleshchev's good lattice with respect to (q, \vec{Q}) which connects $\underline{\emptyset}$ to λ . Now we apply Theorem 4.2, the proposition follows at once. \square

5. EXPLICIT FORMULAS FOR THE NUMBER OF SIMPLE MODULES

In this section, we shall derive explicit formulas for the number of simple modules over the cyclotomic Hecke algebras of type $G(r, p, n)$.

By Lemma 2.4 and [22, (6.2), (6.3)],

$$\begin{aligned} \# \text{Irr}(\mathcal{H}_K(r, p, n)) &= \frac{1}{p} \left\{ \# \text{Irr}(\mathcal{H}_K(r, n)) - \sum_{1 \leq m < p, m|p} N(m) \right\} \\ (5.1) \quad &+ \sum_{1 \leq m < p, m|p} \frac{N(m)}{m} \frac{p}{m}, \end{aligned}$$

where for each integer $1 \leq m, \tilde{m} \leq p$ with $m|p, \tilde{m}|p$,

$$N(\tilde{m}) = \sum_{1 \leq m \leq \tilde{m}, m|\tilde{m}} \mu(\tilde{m}/m) \tilde{N}(m)$$

$$\tilde{N}(m) := \# \{ \lambda \in \mathcal{K}_n \mid h^m(\lambda) = \lambda \},$$

and $\mu(?)$ is the Möbius function. Note that (by [7]) $\#\text{Irr}(\mathcal{H}_K(r, n))$ is explicitly known. Therefore, it suffices to compute $\tilde{N}(m)$. By the discussion in last section, to compute $\tilde{N}(m)$, it suffices to consider the following two cases:

Case 1. $\vec{Q} = (\vec{Q}^{[1]}, \dots, \vec{Q}^{[p]})$, $q^{\mathbb{Z}} \cap \varepsilon^{\mathbb{Z}} = \{1\}$, and for each $1 \leq j \leq p$,

$$\vec{Q}^{[j]} = (\varepsilon^{j-1}q^{v_1}, \dots, \varepsilon^{j-1}q^{v_d}),$$

where $0 \leq v_1, \dots, v_d \leq e-1$ are some integers independent of j .

Case 2. $\vec{Q} = (\vec{Q}^{[1]}, \dots, \vec{Q}^{[k]})$, where $p = d_0k$, q is a primitive $d_0\ell$ -th root of unity, $e = d_0l$, $q^\ell = \varepsilon^k$ is a primitive d_0 -th root of unity, and $1 \leq k < p$ is the smallest positive integer such that $\varepsilon^k \in q^{\mathbb{Z}}$, and for each $1 \leq j \leq k$,

$$\vec{Q}^{[j]} = (\varepsilon^{j-1}q^{v_1}, \dots, \varepsilon^{j-1}q^{v_d}, \varepsilon^{k+j-1}q^{v_1}, \dots, \varepsilon^{k+j-1}q^{v_d}, \dots, \dots, \varepsilon^{(d_0-1)k+j-1}q^{v_1}, \dots, \varepsilon^{(d_0-1)k+j-1}q^{v_d}),$$

where $0 \leq v_1 \leq \dots \leq v_d \leq l-1$ are some integers independent of j .

The following are the second two main results in this paper, which yield explicit formulas for the number of simple modules.

Theorem 5.2. *With the notations and assumptions as in Case 1, let $1 \leq m \leq p$ be an integer such that $m|p$. Let $\vec{Q}^\vee = (q^{v_1}, \dots, q^{v_d})$. If $p \nmid mn$, then $\tilde{N}(m) = 0$; if $p|mn$, then*

$$\tilde{N}(m) = \sum_{n_1 + \dots + n_m = \frac{nm}{p}} \prod_{i=1}^m \left(\#\text{Irr } \mathcal{H}_{q, \vec{Q}^\vee}(d, n_i) \right).$$

where $\mathcal{H}_{q, \vec{Q}^\vee}(d, n_i)$ is the Ariki-Koike algebra with parameters (q, \vec{Q}^\vee) and of size n_i .

Theorem 5.3. *With the notations and assumption as in Case 2, let $1 \leq m \leq p$ be an integer such that $m|p$. Let $a = \gcd(m, k)$, $\tilde{d} = \gcd(\frac{m}{a}, d_0)$. Let q'' be a primitive $\tilde{d}l$ -th root of unity. Let*

$$\vec{Q}^\vee = ((q'')^{v_1}, \dots, (q'')^{v_d}, (q'')^{v_1+l}, \dots, (q'')^{v_d+l}, \dots, \dots, (q'')^{v_1+(\tilde{d}-1)l}, \dots, (q'')^{v_d+(\tilde{d}-1)l}).$$

If $k \nmid na$, then $\tilde{N}(m) = 0$; if $k|na$, then

$$\tilde{N}(m) = \sum_{n_1 + \dots + n_a = \frac{na}{k}} \prod_{i=1}^a \left(\#\text{Irr } \mathcal{H}_{q'', \vec{Q}^\vee}(\tilde{d}d, \frac{\tilde{d}n_i}{d_0}) \right).$$

where $\mathcal{H}_{q'', \vec{Q}^\vee}(\tilde{d}d, \frac{\tilde{d}n_i}{d_0})$ is the Ariki-Koike algebra with parameters (q'', \vec{Q}^\vee) and of size $\frac{\tilde{d}n_i}{d_0}$, and the number $\#\text{Irr } \mathcal{H}_{q'', \vec{Q}^\vee}(\tilde{d}d, \frac{\tilde{d}n_i}{d_0})$ is understood as 0 if $d_0 \nmid \tilde{d}n_i$.

Proof of Theorem 5.2: Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$. We write $\lambda = (\lambda^{[1]}, \dots, \lambda^{[p]})$, where for each $1 \leq j \leq p$,

$$\lambda^{[j]} = (\lambda^{((j-1)d+1)}, \dots, \lambda^{(jd)}).$$

By [11, (4.11)], $\lambda \in \mathcal{K}_n$ if and only for each $1 \leq i \leq p$, $\lambda^{[i]}$ is a Kleshchev d -multipartition with respect to (q, \vec{Q}^\vee) .

By (4.7), we know that

$$h^m(\lambda) = \left(\underbrace{\lambda^{[p-m+1]}, \lambda^{[p-m+2]}, \dots, \lambda^{[p]}}_{m \text{ terms}}, \underbrace{\lambda^{[1]}, \lambda^{[2]}, \dots, \lambda^{[p-m]}}_{p-m \text{ terms}} \right).$$

It is easy to see that $h^m(\lambda) = \lambda$ if and only if

$$\lambda^{[i]} = \lambda^{[i+lm]} \quad \text{for each } 1 \leq l \leq p/m - 1 \text{ and each } 1 \leq i \leq m,$$

from which Theorem 5.2 follows immediately.

We now turn to the proof of Theorem 5.3. We will use the same strategy as in [22, Section 5], where the proof makes use of Naito–Sagaki’s work ([29], [30]).

We keep the notations and assumption as in Case 2. For the moment, we assume that $k = 1$. In other words, $\varepsilon = q^\ell$, q is a primitive pl -th root of unity, $e = pl$ and

$$\vec{Q} = (q^{v_1}, \dots, q^{v_d}, \varepsilon q^{v_1}, \dots, \varepsilon q^{v_d}, \dots, \dots, \varepsilon^{p-1} q^{v_1}, \dots, \varepsilon^{p-1} q^{v_d}),$$

for some integers $0 \leq v_1 \leq \dots \leq v_d \leq l - 1$. We consider the affine Kac–Moody algebra $\mathfrak{g} = \widehat{\mathfrak{sl}}_{pl}$ of type $A_{pl-1}^{(1)}$. Let \mathfrak{h} be the Cartan subalgebra of \mathfrak{g} , let W be the Weyl group of \mathfrak{g} . Let $I := \mathbb{Z}/pl\mathbb{Z}$. Let $\pi : I \rightarrow I$ be the Dynkin diagram automorphism of order p/m defined by $\bar{i} = i + pl\mathbb{Z} \mapsto \bar{i} - \overline{ml} = i - ml + pl\mathbb{Z}$ for any $\bar{i} \in I$. By [13], π induces a Lie algebra automorphism (which is called the diagram outer automorphism) $\pi \in \text{Aut}(\mathfrak{g})$ of order p/m and a linear automorphism $\pi^* \in GL(\mathfrak{h}^*)$ of order p/m . Let $\check{\mathfrak{g}}$ be the corresponding orbit Lie algebra. Then (by [22, (6.4)])

$$\check{\mathfrak{g}} = \begin{cases} \widehat{\mathfrak{sl}}_{ml}, & \text{if } ml > 1, \\ \mathbb{C}, & \text{if } m = l = 1. \end{cases}$$

Let $\check{\mathfrak{h}}$, \check{W} , $\{\check{\Lambda}_i\}_{0 \leq i \leq ml-1}$ denote the Cartan subalgebra, the Weyl group, the set of fundamental dominant weights of $\check{\mathfrak{g}}$ respectively. Let $\widetilde{W} = \{w \in W \mid \pi^* w = w \pi^*\}$. There exists a linear automorphism $P_\pi^* : \check{\mathfrak{h}}^* \rightarrow (\mathfrak{h}^*)^\circ := \{\Lambda \in \mathfrak{h}^* \mid \pi^*(\Lambda) = \Lambda\}$ and a group isomorphism $\Theta : \check{W} \rightarrow \widetilde{W}$ such that $\Theta(\check{w}) = P_\pi^* \check{w} (P_\pi^*)^{-1}$ for each $\check{w} \in \check{W}$. By [13, §6.5], for each $0 \leq i < ml$,

$$P_\pi^*(\check{\Lambda}_i) = \Lambda_i + \Lambda_{i+ml} + \Lambda_{i+2ml} + \dots + \Lambda_{i+(p-m)l} + C\delta,$$

where $C \in \mathbb{Q}$ is some constant depending on π , δ denotes the null root of \mathfrak{g} . Let

$$\check{\Lambda} := \sum_{i=1}^d \sum_{j=1}^m \Lambda_{v_i+(j-1)l}, \quad \Lambda := \sum_{i=1}^d \sum_{j=1}^p \Lambda_{v_i+(j-1)l}.$$

Then it follows that $P_\pi^*(\check{\Lambda}) = \Lambda + C'\delta$, for some $C' \in \mathbb{Q}$.

Let $\varepsilon' := \varepsilon^{p/m}$, which is a primitive m -th root of unity. By Lemma 4.6, we can find a primitive ml th root of unity q' , such that $(q')^l = \varepsilon'$. Let

$$\begin{aligned} \vec{Q}^\vee = & (q^{v_1}, \dots, q^{v_d}, \varepsilon q^{v_1}, \dots, \varepsilon q^{v_d}, \dots, \dots, \\ & \varepsilon^{m-1} q^{v_1}, \dots, \varepsilon^{m-1} q^{v_d}), \end{aligned}$$

By the same argument as in [22, (6.9),(6.10)], we get that

Corollary 5.4. *With the notation as above, there exists a bijection $\eta : \check{\lambda} \mapsto \lambda$ from the set of Kleshchev dm -multipartitions $\check{\lambda}$ of nm/p with respect to (q', \vec{Q}^\vee) onto the set of Kleshchev dp -multipartitions λ of n with respect to (q, \vec{Q}) satisfying $h^m(\lambda) = \lambda$, such that if*

$$\underbrace{(\emptyset, \dots, \emptyset)}_{dm \text{ copies}} \xrightarrow{r_1} \dots \xrightarrow{r_2} \dots \xrightarrow{r_s} \check{\lambda}$$

is a path from $\underbrace{(\emptyset, \dots, \emptyset)}_{dm \text{ copies}}$ to $\check{\lambda}$ in Kleshchev's good lattice with respect to

(q', \vec{Q}^\vee) , where $s := nm/p$, then the sequence

$$\begin{aligned} \underbrace{(\emptyset, \dots, \emptyset)}_{dp \text{ copies}} & \xrightarrow{r_1} \dots \xrightarrow{ml+r_1} \dots \xrightarrow{(p-m)l+r_1} \dots \xrightarrow{r_2} \dots \xrightarrow{ml+r_2} \dots \xrightarrow{(p-m)l+r_2} \dots \\ & \dots \xrightarrow{r_s} \dots \xrightarrow{ml+r_s} \dots \xrightarrow{(p-m)l+r_s} \lambda \end{aligned}$$

defines a path in Kleshchev's good lattice (w.r.t., (q, \vec{Q})) satisfying $h^m(\lambda) = \lambda$. In particular, Theorem 5.3 is valid in the case $k = 1$. That is,

$$\tilde{N}(m) = \# \text{Irr } \mathcal{H}_{q', \vec{Q}^\vee} \left(md, \frac{mn}{p} \right).$$

Now we consider the case where $k > 1$. We keep the notation and assumption as in Case 2. That is, $\vec{Q} = (\vec{Q}^{[1]}, \dots, \vec{Q}^{[k]})$, where $p = d_0 k$, q is a primitive $d_0 \ell$ -th root of unity, $q^\ell = \varepsilon^k$ is a primitive d_0 -th root of unity, $e = d_0 l$, and $1 < k < p$ is the smallest positive integer such that $\varepsilon^k \in q^{\mathbb{Z}}$, and for each $1 \leq j \leq k$,

$$\begin{aligned} \vec{Q}^{[j]} = & (\varepsilon^{j-1} q^{v_1}, \dots, \varepsilon^{j-1} q^{v_d}, \varepsilon^{k+j-1} q^{v_1}, \dots, \varepsilon^{k+j-1} q^{v_d}, \dots, \dots, \\ & \varepsilon^{(d_0-1)k+j-1} q^{v_1}, \dots, \varepsilon^{(d_0-1)k+j-1} q^{v_d}), \end{aligned}$$

for some integers $0 \leq v_1 \leq \dots \leq v_d \leq l-1$. Let $\lambda = (\lambda^{[1]}, \dots, \lambda^{[k]})$, where for each $1 \leq i \leq k$,

$$\lambda^{[i]} = (\lambda^{((i-1)d_0d+1)}, \dots, \lambda^{((i-1)d_0d+d_0d)}).$$

Let $n_i := |\lambda^{[i]}|$ for each $1 \leq i \leq k$. Clearly, $\lambda \in \mathcal{K}_n$ if and only if for each $1 \leq i \leq k$, $\lambda^{[i]} \in \mathcal{K}_{n_i}$, where \mathcal{K}_{n_i} denotes the set of Kleshchev d_0d -multipartitions of n_i with respect to

$$(q, \underbrace{q^{v_1}, \dots, q^{v_d}}_{d \text{ terms}}, \underbrace{\varepsilon^k q^{v_1}, \dots, \varepsilon^k q^{v_d}}_{d \text{ terms}}, \dots, \underbrace{\varepsilon^{(d_0-1)k} q^{v_1}, \dots, \varepsilon^{(d_0-1)k} q^{v_d}}_{d \text{ terms}}).$$

Suppose that $1 \leq a \leq \min\{m, k\}$ is the greatest common divisor of m and k . We define

$$\tilde{\Sigma}(k, m) := \left\{ (\lambda^{[1]}, \dots, \lambda^{[a]}) \vdash \frac{na}{k} \mid \begin{array}{l} \lambda^{[i]} \in \mathcal{K}_{n_i}, (\mathbf{h}')^{m/a}(\lambda^{[i]}) = \lambda^{[i]}, \\ \forall 1 \leq i \leq a, \sum_{i=1}^a n_i = \frac{na}{k} \end{array} \right\},$$

$$\tilde{N}(k, m) := \#\tilde{\Sigma}(k, m),$$

$$\tilde{\Sigma}(m) := \{\lambda \in \mathcal{K}_n \mid \mathbf{h}^m(\lambda) = \lambda\}.$$

where \mathbf{h}' is the same as in Proposition 4.10.

With the Proposition 4.10 in mind, it is easy to see that the same argument in the proof of [22, Lemma 6.16] proves the following result.

Lemma 5.5. *The map which sends $\lambda = (\lambda^{[1]}, \dots, \lambda^{[k]})$ to $\bar{\lambda} := (\lambda^{[1]}, \dots, \lambda^{[a]})$ defines a bijection from the set $\tilde{\Sigma}(m)$ onto the set $\tilde{\Sigma}(k, m)$.*

Let $\tilde{d} := \gcd(d_0, \frac{m}{a})$. Note that $(\mathbf{h}')^{d_0}(\lambda^{[i]}) = \lambda^{[i]}$ for each $1 \leq i \leq k$. Therefore,

$$(5.6) \quad (\mathbf{h}')^{m/a}(\lambda^{[i]}) = \lambda^{[i]} \text{ if and only if } (\mathbf{h}')^{\tilde{d}}(\lambda^{[i]}) = \lambda^{[i]}.$$

Note that we have just proved Theorem 5.3 in the case where $k = 1$, it is now easy to see that Theorem 5.3 in the case where $k > 1$ follows directly from Proposition 4.10, Corollary 5.4, Lemma 5.5 and (5.6). This completes the proof of Theorem 5.3 in all cases.

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Appendix by Xiaoyi Cui

Let R be a commutative ring with identity 1_R . Let B be a finitely generated R -free R -algebra. Let A be a $\mathbb{Z}/r\mathbb{Z}$ -graded algebra over the subalgebra B with grading $A = \bigoplus_{j=0}^{r-1} a^j B$ for a unit $a \in A$, $aB = Ba$, $a^r \in B$. Furthermore, we assume that $r \cdot 1_R$ is a unit in R , and R contains a primitive r th root of unity ϵ .

Let $\{b_i\}_{i=1}^s$ be an R -basis of B . Then, the set $\{b_i a^j\}_{1 \leq i \leq s, 0 \leq j < r}$ is an R -basis of A . Furthermore, the set

$$\left\{ b_{i_1} a^{i_2} \otimes_B a^{i_3} \mid 1 \leq i_1 \leq s, 0 \leq i_2, i_3 < r \right\}$$

is an R -basis of $A \otimes_B A$.

Let σ be the automorphism of A which is defined by

$$a^j x \mapsto \epsilon^j a^j x, \quad \forall x \in B, j \in \mathbb{Z}.$$

For each integer j with $0 \leq j < r$, recall that $A^{\sigma^j} = A$ as R -module, the left A -action on A^{σ^j} is given by the usual left multiplication, while the right A -action on A^{σ^j} is given by the usual right multiplication twisted by σ^j . To avoid confusion, we use the symbol $(b_{i_1} a^{i_2})_{(j)}$ to denote the element $b_{i_1} a^{i_2}$ in A^{σ^j} . Then the elements

$$(b_{i_1} a^{i_2})_{(j)}, \quad \text{where } i_1 \in \{1, 2, \dots, s\}, i_2, j \in \{0, 1, \dots, r-1\},$$

form an R -basis of $\bigoplus_{j=0}^{r-1} A^{\sigma^j}$.

The conclusion of the following fact is contained in the proof of [G, Proposition 2.2]. However, the argument given by the proof of [G, Proposition 2.2] contains a gap. That is, in the 10th line of Page 527, Genet's claim about the determinant of the representing matrix is generally false. In fact, it is a quite nontrivial job to calculate the the determinant of that representing matrix, as one can see from the following proof.

Fact: Let $\varphi : A \otimes_B A \rightarrow \bigoplus_{j=0}^{r-1} A^{\sigma^j}$ be the R -linear homomorphism defined on basis by

$$\varphi(b_{i_1} a^{i_2} \otimes_B a^{i_3}) = \bigoplus_{j=0}^{r-1} \left(\epsilon^{ji_3} (b_{i_1} a^{i_2+i_3})_{(j)} \right),$$

where $i_1 \in \{1, 2, \dots, s\}$, $i_2, i_3 \in \{0, 1, \dots, r-1\}$. Then φ is an (A, A) -bimodule isomorphism.

Proof. It is easy to verify that φ is an (A, A) -bimodule homomorphism. Therefore, it remains to show that φ is an R -linear isomorphism.

For any integers i_1, i_2, i_3 with $1 \leq i_1 \leq s, 0 \leq i_2, i_3 \leq r-1$, we define

$$X_{i_3 sr + i_2 s + i_1} := b_{i_1} a^{i_2} \otimes_B a^{i_3}.$$

Then the set $\{X_1, X_2, \dots, X_{sr^2}\}$ is an ordered R -basis of $A \otimes_B A$.

For any integers k_1, k_2, k_3 with $1 \leq k_2 \leq s, 0 \leq k_1, k_3 \leq r-1$, we define

$$Y_{k_1 sr + k_3 s + k_2} := (b_{k_2} a^{k_3})_{(k_1)}.$$

Then the set $\{Y_1, Y_2, \dots, Y_{sr^2}\}$ is an ordered R -basis of $\oplus_{j=0}^{r-1} A^{\sigma^j}$.

We want to compute the determinant of the representing matrix of φ with respect to the ordered X -basis and the ordered Y -basis. Suppose that

$$\varphi(X_1, X_2, \dots, X_{sr^2}) = (Y_1, Y_2, \dots, Y_{sr^2})M,$$

where M is an $sr^2 \times sr^2$ matrix. For any integers i, j with $0 \leq i, j \leq r-1$, we use $X \downarrow_{i_3=i}$ (resp., $Y \downarrow_{k_1=j}$) to denote the naturally ordered basis elements

$$\{X_{isr+i_2s+i_1} \mid 1 \leq i_1 \leq s, 0 \leq i_2 \leq r-1\}$$

$$\left(\text{resp., } \{Y_{jsr+k_3s+k_2} \mid 0 \leq k_2, k_3 \leq r-1\} \right).$$

We use ι_i to denote the embedding from the free R -submodule spanned by elements in $X \downarrow_{i_3=i}$ into $A \otimes_B A$; we use p_j to denote the natural projection from $\oplus_{t=0}^{r-1} A^{\sigma^t}$ onto A^{σ^j} (i.e., the free R -submodule spanned by the elements in $Y \downarrow_{k_1=j}$).

Let

$$\varphi \downarrow_{i_3=i, k_1=j} := p_j \circ \varphi \circ \iota_i.$$

Note that $X \downarrow_{i_3=i}$ (resp., $Y \downarrow_{k_1=j}$) is a consecutive part of $(X_1, X_2, \dots, X_{sr^2})$ (resp., of $(Y_1, Y_2, \dots, Y_{sr^2})$). Therefore, we can partition M as follows.

$$M = \begin{pmatrix} M_{0,0} & M_{0,1} & \cdots & M_{0,r-1} \\ M_{1,0} & M_{1,1} & \cdots & M_{1,r-1} \\ \vdots & \vdots & \vdots & \vdots \\ M_{r-1,0} & M_{r-1,1} & \cdots & M_{r-1,r-1} \end{pmatrix},$$

where for each pair of integers (i, j) with $0 \leq i, j \leq r-1$, $M_{j,i}$ is the representing matrix of $\varphi \downarrow_{i_3=i, k_1=j}$ with respect to the ordered basis $X \downarrow_{i_3=i}$ and $Y \downarrow_{k_1=j}$. That is

$$\varphi \downarrow_{i_3=i, k_1=j} (X \downarrow_{i_3=i}) = (Y \downarrow_{k_1=j}) M_{j,i}.$$

Note that each $M_{j,i}$ is an $rs \times rs$ matrix. We claim that

(1) for each pair of integers (i, j) with $0 \leq i, j \leq r-1$,

$$M_{j,i} = e^{ji} M_{0,i}.$$

(2) for each integer i with $0 \leq i \leq r-1$, $M_{0,i} = (M_{0,1})^i$.

In fact, claim (1) follows directly from the definition of φ and our ordering of the X basis and the Y basis. It suffices to prove the claim (2).

For any integer i with $0 \leq i \leq r-1$, we set

$$\varphi_i := \varphi \downarrow_{i_3=i, k_1=0}: b_{i_1} a^{i_2} \otimes a^i \mapsto (b_{i_1} a^{i_2+i})_{(0)}.$$

We identify the free R -submodule of $A \otimes_B A$ spanned by elements in $X \downarrow_{i_3=i}$ with $A^{\sigma^0} = A$ via

$$b_{i_1} a^{i_2} \otimes a^i \leftrightarrow (b_{i_1} a^{i_2})_{(0)}, \text{ for any integers } 1 \leq i_1 \leq s, 0 \leq i_2 \leq r-1.$$

With the above identification in mind, it is easy to see that

$$\varphi_i = (\varphi_1)^i.$$

As a result, $M_{0,i} = (M_{0,1})^i$. This proves claim (2).

Therefore,

$$\begin{aligned} M &= \begin{pmatrix} I & M_{0,1} & (M_{0,1})^2 & \cdots & (M_{0,1})^{r-1} \\ I & \epsilon M_{0,1} & (\epsilon M_{0,1})^2 & \cdots & (\epsilon M_{0,1})^{r-1} \\ I & \epsilon^2 M_{0,1} & (\epsilon^2 M_{0,1})^2 & \cdots & (\epsilon^2 M_{0,1})^{r-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I & \epsilon^{r-1} M_{0,1} & (\epsilon^{r-1} M_{0,1})^2 & \cdots & (\epsilon^{r-1} M_{0,1})^{r-1} \end{pmatrix} \\ &= \begin{pmatrix} I & I & (I)^2 & \cdots & (I)^{r-1} \\ I & \epsilon I & (\epsilon I)^2 & \cdots & (\epsilon I)^{r-1} \\ I & \epsilon^2 I & (\epsilon^2 I)^2 & \cdots & (\epsilon^2 I)^{r-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I & \epsilon^{r-1} I & (\epsilon^{r-1} I)^2 & \cdots & (\epsilon^{r-1} I)^{r-1} \end{pmatrix} \times \\ &\quad \begin{pmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & M_{0,1} & 0 & \cdots & 0 \\ 0 & 0 & M_{0,1}^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (M_{0,1})^{r-1} \end{pmatrix} \\ &= V_r \times D, \end{aligned}$$

where I is the $rs \times rs$ identity matrix,

$$\begin{aligned} V_r &= \begin{pmatrix} I & I & I & \cdots & I \\ I & \epsilon I & \epsilon^2 I & \cdots & \epsilon^{r-1} I \\ I & \epsilon^2 I & \epsilon^4 I & \cdots & \epsilon^{2(r-1)} I \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I & \epsilon^{r-1} I & \epsilon^{2(r-1)} I & \cdots & \epsilon^{(r-1)^2} I \end{pmatrix}; \\ D &= \begin{pmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & M_{0,1} & 0 & \cdots & 0 \\ 0 & 0 & M_{0,1}^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (M_{0,1})^{r-1} \end{pmatrix}. \end{aligned}$$

Hence, $\det M = \det V_r \det D = \det V_r (\det M_{0,1})^{r(r-1)/2}$. To show that φ is an isomorphism, it suffices to show that $\det M$ is a unit in R . Therefore, it suffices to show that both $\det V_r$ and $\det M_{0,1}$ are units in R .

By assumption, a is invertible in A , which implies that the elements in $\{b_1a^r, b_2a^r, \dots, b_sa^r\}$ are R -linear independent. Also, the condition that $a^r \in B$ implies that there exists a matrix $C = (C_{i,j})_{s \times s} \in M_{s \times s}(R)$, such that $(b_1a^r, b_2a^r, \dots, b_sa^r) = (b_1, b_2, \dots, b_s)C$.

By the condition that a is invertible in A we deduce that C is invertible in $M_{s \times s}(R)$. In particular, $\det C$ is invertible in R , i.e., a unit in R . By direct calculation, we know that

$$M_{0,1} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & C \\ I & 0 & 0 & \dots & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \end{pmatrix},$$

where each 0 denotes an $s \times s$ zero matrix, each I denote an $s \times s$ identity matrix. As a consequence, $\det M_{0,1} = (-1)^{(r-1)s^2} \det C$ is invertible in R .

It remains to show that $\det V_r$ is invertible in R . In fact,

$$\begin{aligned} & \det V_r \\ = & \begin{vmatrix} I & I & I & \dots & I \\ 0 & (\epsilon - 1)I & (\epsilon^2 - 1)I & \dots & (\epsilon^{r-1} - 1)I \\ 0 & (\epsilon^2 - \epsilon)I & (\epsilon^4 - \epsilon^2)I & \dots & (\epsilon^{2(r-1)} - \epsilon^{r-1})I \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (\epsilon^{r-1} - \epsilon^{r-2})I & (\epsilon^{2(r-1)} - \epsilon^{2(r-2)})I & \dots & (\epsilon^{(r-1)^2} - \epsilon^{(r-1)(r-2)})I \end{vmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} & \det V_r \\ = & \begin{vmatrix} (\epsilon - 1)I & (\epsilon^2 - 1)I & \dots & (\epsilon^{r-1} - 1)I \\ (\epsilon^2 - \epsilon)I & (\epsilon^4 - \epsilon^2)I & \dots & (\epsilon^{2(r-1)} - \epsilon^{r-1})I \\ \vdots & \vdots & \vdots & \vdots \\ (\epsilon^{r-1} - \epsilon^{r-2})I & (\epsilon^{2(r-1)} - \epsilon^{2(r-2)})I & \dots & (\epsilon^{(r-1)^2} - \epsilon^{(r-1)(r-2)})I \end{vmatrix} \\ = & ((\epsilon - 1)(\epsilon^2 - 1) \dots (\epsilon^{r-1} - 1))^{rs} \begin{vmatrix} I & I & \dots & I \\ \epsilon I & \epsilon^2 I & \dots & \epsilon^{r-1} I \\ \epsilon^2 I & \epsilon^4 I & \dots & \epsilon^{2(r-1)} I \\ \vdots & \vdots & \vdots & \vdots \\ \epsilon^{r-2} I & \epsilon^{2(r-2)} I & \dots & \epsilon^{(r-1)(r-2)} I \end{vmatrix} \\ = & \left(\epsilon^{(r-1)(r-2)/2} \prod_{t=1}^{r-1} (\epsilon^t - 1) \right)^{rs} \begin{vmatrix} I & I & \dots & I \\ I & \epsilon I & \dots & \epsilon^{r-2} I \\ I & \epsilon^2 I & \dots & \epsilon^{2(r-2)} I \\ \vdots & \vdots & \vdots & \vdots \\ I & \epsilon^{r-2} I & \dots & \epsilon^{(r-2)^2} I \end{vmatrix} \end{aligned}$$

$$= \left(\epsilon^{(r-1)(r-2)/2} \prod_{t=1}^{r-1} (\epsilon^t - 1) \right)^{rs} \det V_{r-1}.$$

Now by an easy induction argument, it is easy to see that

$$\det V_r = \left(\epsilon^{\sum_{a=1}^{r-2} \frac{a(a+1)}{2}} \prod_{b=1}^{r-1} \prod_{t=1}^b (\epsilon^t - 1) \right)^{rs}.$$

Note that $r = \prod_{1 \leq j \leq r-1} (1 - \epsilon^j)$. By assumption, r is a unit in R . It follows that $\det V_r$ must be an invertible element in R . This completes the proof. \square

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